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CALCULUS.

281. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that if n be a positive integer greater than unity,

$$\operatorname{Log} n=(1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots+\frac{1}{n-1}+\frac{1}{2n})-C+\frac{B_1}{2n^2}-\frac{B_2}{4n^4}+\frac{B_3}{6n^6}-\frac{B_4}{8n^8}+\ldots(1).$$

REMARK.—This development may be obtained by employing the formula given by the proposer in *Annals of Mathematics*, second series, Vol. 5, No. 4, July, 1904, but other proofs are also desired. It will be observed that (1) offers a ready method of evaluating C, which is remarkably simple and very rapidly convergent if $n>$ or $=10$. Compare with method given by Mr. Bromwitch in *Messenger of Mathematics*, October, 1906.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

In Art. 304, Todhunter's *Integral Calculus*, we find the following result deduced:

$$\Sigma\phi(x)=C+\frac{1}{h}\int\phi(x)dx-\tfrac{1}{2}\phi(x)+B_1\phi'(x)\frac{h}{2}-B_2\phi'''(x)\frac{h^3}{4!}$$

$$+B_3\phi^{\mathrm{v}}(x)\frac{h^5}{6!}-B_4\phi^{\mathrm{vii}}(x)\frac{h^7}{8!}+\ldots$$

Let $h=1$, $\phi(x)=1/n$. Then $\phi'(x)=-1/n^2$, $\phi'''(x)=-3!/n^4$, $\phi^{\mathrm{v}}(x)=-5!/n^6$, $\phi^{\mathrm{vii}}(x)=-7!/n^8$.

$$\Sigma\phi(x)=1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n-1}.$$

$\therefore\ \Sigma\left(\frac{1}{n}\right)=C+\log n-\frac{1}{2n}-\frac{B_1}{2n^2}+\frac{B_2}{4n^4}-\frac{B_3}{6n^6}+\frac{B_4}{8n^8}-\ldots$ whence the given result.

II. Solution by C. N. SCHMALL, New York City.

Take the well known relation

$$\Gamma(x+2)=(x+1)\Gamma(x+1)\ldots(1)$$

and take logarithms of both sides,

$$\log\Gamma(x+2)=\log\Gamma(x+1)+\log(x+1)\ldots(2).$$

Now define a function $\phi(x)$ by the relation

$$\phi(x)=\frac{d}{dx}\log\Gamma(1+x)\ldots(3).$$

Differentiating equation (2) we have

$$\phi(x+1)=\phi(x)+\frac{1}{x+1}\ldots(4).$$

$\therefore \phi(x)=\phi(x+1)-\frac{1}{x+1}$. Again, $\phi(x+2)=\phi[(x+1)+1]$, and by (4),

$$\phi[(x+1)+1]=\phi(x+1)+\frac{1}{(x+1)+1}=\phi(x+1)+\frac{1}{x+2}=\phi(x)+\frac{1}{x+1}+\frac{1}{x+2}$$

Transposing, $\phi(x)=\phi(x+2)-\frac{1}{x+1}-\frac{1}{x+2}$. Hence, by induction, we have

$$\phi(x)=\phi(x+n)-\left[\frac{1}{x+1}+\frac{1}{x+2}+\ldots+\frac{1}{x+n}\right]\ldots(5),$$

where n is an integer. Put $x=0$, then

$$\phi(n)=-C+1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\frac{1}{n}\ldots(6),$$

where $-C=\phi(0)$. Now (6) may be written

$$\phi(n)=-C+\int_0^1(1+z+z^2+\ldots+z^{n-1})\,dz=-C+\int_0^1\frac{(1-z^n)}{1-z}dz\ldots(7).$$

This defines $\phi(n)$ for integral values of n. Again, by (7),

$$\phi(n+1)-\phi(n)=\int_0^1\frac{(1-z^{n+1})\,dz}{1-z}-\int_0^1\frac{(1-z^n)\,dz}{1-z}=\int_0^1\frac{(z^n-z^{n+1})\,dz}{1-z}$$

$$=\int_0^1\frac{z^n(1-z)}{1-z}dz=\int_0^1 z^n dz=\frac{1}{n+1}.$$

Putting x for n, this becomes

$$\phi(x+1)-\phi(x)=\frac{1}{x+1}$$

which agrees with (4). Hence n may also be a fraction; and we have, in general, from (7),

$$\phi(x) = -C + \int_0^1 \frac{(1-z^x)}{1-z} dz = -C + \int_0^1 (1-z^x)(1+z+z^2+\ldots) dz$$

$$= -C + \frac{x}{x+1} + \tfrac{1}{2}\frac{x}{x+2} + \tfrac{1}{3}\frac{x}{x+3} + \ldots \text{ (8).}$$

Replace $\phi(x)$ by its value from (3), and integrate,

$$\log \Gamma(1+x) = -Cx + x - \log(1+x) + \tfrac{1}{2}x - \log(1+\tfrac{1}{2}x) + \tfrac{1}{3}x - \log(1+\tfrac{1}{3}x) + \ldots$$

Raise e to the power of each side,

$$\Gamma(1+x) = e^{-cx} . \frac{e^x}{1+x} . \frac{e^{\frac{1}{2}x}}{1+\frac{1}{2}x} . \frac{e^{\frac{1}{3}x}}{1+\frac{1}{3}x} \ldots \text{ (9).}$$

Putting $x=1$, $1 = e^{-c} . \tfrac{1}{2}e . \frac{2e^{1/2}}{2} . \frac{3e^{1/3}}{4} . \frac{4e^{1/4}}{5} \ldots$ Taking logarithms,

$$0 = -C + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + (1/n) - \log(n+1).$$

$\therefore \log(n+1) = -C + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + (1/n) \equiv \phi(n)$ [by (6)], when n becomes infinite. Hence, for a very large value of n we have, approximately,

$$\phi(n) = \log(n+1) = \log n + \log(n+1) - \log n$$

$$= \log n + \log(1 + \frac{1}{n}) = \log n + \frac{1}{n} - \frac{1}{2n^2} + \ldots.$$

Hence, $\log n = \log(n+1) - \frac{1}{n} + \frac{1}{2n^2} - \ldots + .. = -C + 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots$

$$+ \frac{1}{n-1} + \frac{1}{n} - \frac{1}{n} + \frac{1}{2n^2} - \ldots + \ldots$$

i. e., $\log n = (1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \frac{1}{n-1} - C + \frac{1}{2n^2} - \ldots + \ldots$

We may obtain C in the form of a definite integral, as follows:

$$\phi(x) = \frac{d}{dx} \log \Gamma(1+x) \text{ (by definition) } = \frac{d}{dx}\Gamma(1+x)/\Gamma(1+x).$$

$$\therefore \phi(x)\Gamma(1+x)=\frac{d}{dx}\Gamma(1+x)=\frac{d}{dx}\int_0^\infty z^x e^{-z}dz=\int_0^\infty z^x e^{-z}\log z\,dz.$$

Put $x=0$; and since $\Gamma(1)=1$, we have

$$\phi(0)=\int_0^\infty e^{-z}\log z\,dz. \quad \therefore C=-\phi(0)=\int_0^\infty e^{-z}\log\left(\frac{1}{z}\right)dz$$

which gives Euler's constant.

MECHANICS.

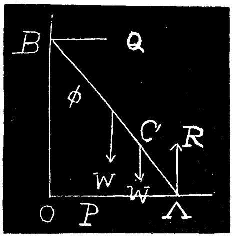
234. Proposed by C. N. SCHMALL, 89 Columbia Street, New York City.

A ladder is placed with one end resting against a smooth wall and making with it an angle ϕ. Also, the roughness of the ground prevents it from slipping. A man weighing as much as the ladder ascends to the top. Taking μ as the coefficient of friction, prove:

(a) The ladder will slip before he gets to the top if $\phi>\tan^{-1}4\mu/3$.

(b) If the ascent be feasible, there will be three times as much friction when he is at the top as when he is at the bottom, (See Jeans' *Theoretical Mechanics*, p. 47.)

I. Solution by J. SCHEFFER, A. M., Kee Mar College, Hagerstown, Md.



Let $AB=c$ be the ladder; let some position of the man be at C, AC being $=a$. Let the reaction at A and B be, respectively, $=P$ and Q, and the ladder be uniform, that is, its weight W acting at the mid-point of AB; the friction $P\mu$ acts along AO. Resolving along AO and BO, we have $Q=P\mu$, $P=2W$, and the equation of the moments with reference to A, $W(a+\frac{1}{2}c)\sin\phi=Qc\cos\phi=2W\mu c\cos\phi$; whence $\tan\phi=\frac{4c\mu}{2a+c}$. For $a=c$, $\tan\phi=\frac{4\mu}{3}$, $Q=\frac{1}{2}W\tan\phi$ $=\frac{2}{3}W\mu.P\mu=2W\mu$.

II. Solution by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

Let $2l$ be the length of the ladder, R the pressure on the ground, R' the pressure on the wall. Assuming that the man is at the top and that the ladder is about to slip, equating the sums of the vertical and horizontal forces to zero and taking moments about the foot of the ladder, we find

$$R=2W,\quad F=\mu R=R',\quad R'.2l\cos\phi=Wl\sin\phi+W.2l\sin\phi=3Wl\sin\phi.$$

Whence, $2R'=3W\tan\phi$. But $R'=2\mu W$. Therefore, $4\mu=3\tan\phi$ or $\phi=\tan^{-1}(4\mu/3)$. A larger value of ϕ makes $R'>F$, and the man cannot reach the top before slipping begins.